

THE DEFINITIONS OF SPATIAL QUANTITIES IN ELEMENTARY CURRICULUM MATERIALS

Nicholas J. Gilbertson
Winona State University
ngilbertson@winona.edu

Jia He
Utah Valley University
jia.he@uvu.edu

V. Rani Satyam
Michigan State University
satyamvi@msu.edu

John P. Smith III
Michigan State University
stehrery@msu.edu

Eryn M. Stehr
Michigan State University
jsmith@msu.edu

Research indicates that U.S. students have misunderstandings in measurement, and one reason is the highly procedural focus of elementary curricula. Because the definitions of quantities are basic conceptual content, we examined definitional-type expressions (related to formal mathematical definitions) for spatial quantities (length, area and volume) within and across elementary curricula with special attention to the inclusion of units, quantification, and exhausting space. We found inconsistencies in the ways these conceptual aspects were presented. In some cases, the inclusion of particular conceptual content depended on the type of quantity, (e.g. “exhausting space” was strongly emphasized in area measure, but less so in length and volume).

Keywords: Curriculum, Elementary School Education, Measurement

Background

Definitions hold a central place in mathematics. They are distinguished from theorems because definitions cannot be proven true or false, and they are different from axioms because definitions can be contested (Kobiela & Lehrer, 2015). *The Common Core State Standards for Mathematics* indicates in the *Attend to Precision* standard that students should, “...use clear definitions in discussion with others and in their own reasoning” (NGA-CCSSO, 2010, p. 7). The practice of defining is potentially rich as a classroom activity (Gilbertson, 2015); yet, many students encounter definitions as simply being presented by the teacher or their textbook. In some ways this approach is sensible, given the inherent messiness of involving students in the defining process to create precise and meaningful definitions which are both mathematically accurate and useful for testing whether a mathematical object fits the definition or not.

Complicating the issue is that some mathematical ideas span multiple grade levels, meaning that students may have multiple exposures to definitions of the same concept. This creates a dilemma particularly for textbook authors because they must maintain consistency, while using language that is age-appropriate and meaningful to students based on their mathematical experiences. The definition of a mathematical function, for example, might look quite different to a middle school student than a student in an abstract algebra course. *Consistency* is not meant to imply the definitions are identical, rather that the definitions convey similar meaning noting important characteristics of the object being defined. This raises the question as to how fundamental mathematical ideas are conveyed to students across grade levels, and the extent to which these are done consistently or inconsistently.

One approach to study this phenomenon would be to study how teachers define mathematical ideas in their own classrooms. We chose instead to study written curriculum materials because these materials comprise the main mediating tool between the ideas of the broader mathematical community and the ideas developed in classroom interactions. Although one limitation of studying written materials is the variation of implementation across classrooms (Tarr et al., 2008), written curriculum plays an important role in what teachers teach, the opportunities they present for

mathematical sense-making (Stein, Remillard, & Smith, 2007) and their profound impact on K-12 classroom instruction (McCrary, Francis, & Young, 2008).

Instead of studying definitions across content domains, we selected one particular topic and grade band with documented issues of student learning to investigate the extent to which definitions could possibly support or hinder opportunities for student understanding. We chose to study definitions of spatial measurement at the elementary level because educational researchers (e.g., Kamii & Kysh, 2006) and national assessments (e.g., NAEP) have indicated deficiencies in elementary students' conceptual understanding of spatial measurement such as confusing the meaning of area and perimeter (e.g., Bamberger & Oberdorf, 2010; Barrett & Clements, 2003; Woodward & Byrd, 1983) amongst many other issues. Additionally, the lack of clarity about the quantity to be measured and what constitutes a measure of that quantity significantly increases the challenges that students face in understanding the measurement process (Smith, Males, Dietiker, Lee, & Mosier, 2013). Curricular analyses of spatial measurement (e.g. Smith et al., 2013), have argued that deficiencies in written curriculum materials may be a contributing factor to these conceptual deficiencies. Since definitions are central to describing the meaning of terms, it raises the question as to how definitions can be presented to contribute in some way to students' conceptual issues.

Purpose and Research Questions

The purpose of this study is to describe opportunities for students to understand definitions of length, area, and volume in written curriculum materials, with specific attention to the consistencies and inconsistencies across grade levels and materials. We use the term, *definitional-type expression* [DTE] to designate any written description of a term that aims to support students in understanding a term's definition. These include all explicit definitions in the text along with other expressions that describe or delimit a quantity or its measure. In elementary textbooks in particular, it is not always the case that formal definitions appear as explicitly as one might see in higher grade levels (e.g. "The definition of a square is..."). Thus, our broader interpretation of definitions includes expressions which aim to define or support defining, but may not be explicitly indicated by the materials as a "definition." Specifically, for spatial measurement, we define DTEs as (a) expressions of meaning for length/area/volume as *spatial characteristics*, (b) expressions of meaning of length/area/volume as *measure* of characteristics, or (c) expressions which clarify the meanings of spatial characteristics or their measures (e.g., "area is not perimeter"). Framing our inquiry were the following questions:

- *What characteristics of DTEs are found in elementary curricula across spatial measurement topics, e.g., length, area, and volume?*
- *What similarities/differences exist across curricula and measures with respect to DTEs?*

Methods and Data Sources

Textbooks chosen for this study were from three elementary textbook series (Grades K-5): (a) The University of Chicago School Mathematics Project's *Everyday Mathematics* (2007)—henceforth *EM*, (b) Scott Foresman/Addison Wesley's *Mathematics*, Michigan edition (Charles, Crown, & Fennell, 2008)—*SFAW*, and (c) Saxon Publishers' *Saxon Math* (Larson, 2004)—*Saxon*. Selection of the textbooks was based on a large market share (e.g., *EM* and *SFAW*) and variance in features of textbooks (e.g., *EM* is Standards-based and *Saxon* is unique in its direct instruction approach). We explored DTEs for *length* in grades K through 3, *area* in K through 4, and *volume and capacity* in K through 5. The coding stopped at certain grades (e.g., length at grade 3) because the central processes, concepts, and tools for the measurement (e.g., length) had been presented by that grade level.

Four researchers divided into pairs with one pair examining DTEs of volume and another pair examining DTEs of area and length. After comparing and discussing characteristics of DTEs in

length, area, and volume, we agreed on a final list of emergent DTE characteristics: (a) dimensionality, (b) units, (c) continuous versus discrete materials, (d) exhausting the space, (e) specific shapes, (f) boundary, (g) quantification (e.g., counting, applying a formula), (h) tool use, and (i) space. Each pair analyzed each group of DTEs for the listed characteristics and resolved disagreements together or brought to the whole research team for resolution. Typically, the coded unit of curricular content was either a single full sentence, or a clause of a sentence.

Table 1: Analytical Framework for Coding Definitional-Type Expressions in Measurement

Aspects	Coding Questions
Dimensionality	Is dimensionality explicitly stated (i.e. terms such as “3-dimensional” appear in the text)? If yes, what dimensionality? (e.g. 1, 2 or 3 dimensions)
Units	Are units of measurement indicated (e.g. gallons, cm^2 , feet)? If yes, which units?
Continuous vs. Discrete	What type of substance/object fills the space being measured (e.g., a continuous quantity of liquid or a discrete number of linking cubes)?
Exhausting the Space	Is there mention of the need to completely fill, cover or tile the space (e.g. cover the surface with square tiles without gaps and overlaps)?
Specific shape	Is the measure referenced for a specific shape or object (e.g. a rectangular solid, a triangle)?
Boundary	Is there mention of end-points or other enclosure of the space (e.g. a length measure defined as the distance between two points)?
Quantification	Is there mention of counting, layering, reading a tool or applying a formula (e.g. the volume of an object is the number of cubic units)?
Procedural tool use	Is a specific measurement method using a tool and/or unit indicated (e.g. using a ruler to measure length)?
Space	Is the word "space" explicitly mentioned?

While all these categories potentially contribute to students’ understanding of measurement concepts via definitional expressions, we focus our attention on three categories that seem to be most relevant for students’ meaningfully understanding measurement. Specifically, we focus on the categories: (b) *units*, (d) *exhausting the space*, and (g) *quantification*. These three categories are central to understanding quantities and their measures; as any measure of space involves a choice of a unit, and that the measure is a quantification of the units that completely exhaust the space. Results for this paper will focus primarily on these three categories. Table 2 (below) describes examples of coding for these categories.

Table 2: Analytical Framework Used to Code Aspects of Measurement in DTEs

Aspects	Questions and Examples
Units	<p>Are units of measurement indicated? If yes, which units?</p> <p>Yes:... <i>measure in gallons</i>... (Volume: SFAW, gr. 5, p. SRB157)</p> <p>Yes:... <i>number of unit squares</i>... (Area: EM, gr. 4, p. 671)</p> <p>Yes:... <i>unit of measure will be a linking cube</i>... (Length: Saxon, gr. 1, 35-2-4)</p> <p>No: ... <i>measure how much space there is</i>... (Length: EM, gr. 3, p. SRB2)</p>
Exhausting the Space	<p>Is there mention of the need to completely fill, cover or tile the space?</p> <p>Yes:... <i>fill the space taken up by the object</i>... (Volume: EM, gr. 4, p. 868)</p> <p>Yes:... <i>cover the surface without overlaps and/or gaps</i>... (Area: EM, gr. 4, p. 671)</p> <p>Yes:... <i>must be marked off "end to end," without leaving spaces</i>... (Length: EM, gr. 1, p. 282)</p> <p>No: ... <i>To find the area, count the squares inside the shape</i>. (Area: SFAW, gr. 2, p. 351A)</p>
Quantification	<p>Is there mention of counting, layering, reading a tool or applying a formula (specification of procedure is not necessary)?</p> <p>Yes:... <i>by counting how many</i>... (Volume: SFAW, gr. 5, p. 610)</p> <p>Yes:... <i>count the number of tiles</i>... (Area: Saxon, gr. 3, 88-5)</p> <p>Yes:... <i>count the cubes</i>... (Length: SFAW, gr. K, p. 139)</p> <p>No: ... <i>area is the measurement inside</i>... (Area: SFAW, gr. 4, p. 485)</p>

Results

We provide percentage and frequency distributions for DTEs across curricula for length, area, and volume, and summarize findings for each quantity (e.g., length) regarding curricular attention to each characteristic (e.g., units). Note that DTEs could (and did) address multiple characteristics. We provide percentage distributions representing the proportion of DTEs that mention a characteristic across grades for each curriculum (i.e., *SFAW*, *Saxon*, and *EM*) and for each measurement (i.e., length, area and volume) in Figures 1, 2, and 3 below. Over 200 DTEs were found across the three curricula and three measures, with few for length (20), more for area (60) and even more for volume (131). Thus, students have more opportunities to learn definitions for spatial quantities as dimensionality increases, which may be a result of having more procedures as this occurs (e.g. for volume one can count cubes, using various multiplication formulas, and pour liquid in the case of liquid volume).

Across curricula and measures, many DTEs failed to include all three components in any one expression. Additionally, there was inconsistency within each measure and curriculum as to whether the inclusion of units, the inclusion of exhausting space, or the inclusion of quantification was necessary in defining the measure. This was generally true with the exception of *Saxon*. One exception occurred in their description of length measure, where all DTEs included reference to a unit and quantification. The other exception occurred in *Saxon*'s exclusion of needing to specify exhausting space in length and volume.

This last result points to a more general issue of how some curricula placed increased emphasis on particular conceptual knowledge based on the measure. For example, the DTEs in *EM* placed stronger emphasis exhausting space in length and area (greater than 70% of DTEs in both cases), but de-emphasized this conceptual aspect in volume (less than 20% of DTEs). *SFAW* had variation as well in the exhausting space category, with nearly 70% of DTEs in area compared to less than 15% in volume. In the case of emphasizing a particular conceptual aspect in length, this may be attributed to the fact that length is typically emphasized in earlier grades before area and volume, meaning that the development of the concept is necessarily deeper for length compared to the other two measures.

It does however, raise the question as to whether students are able to sufficiently connect conceptual aspects across measures, namely that unitizing, quantification, and exhausting space are essential for all measures, not just the particular ones emphasized in the curriculum.

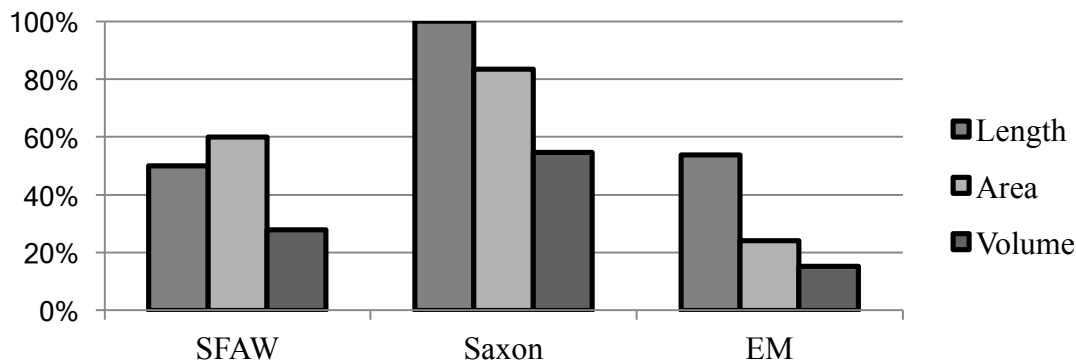


Figure 1. Percentage of DTEs that reference "Units" of measurement.

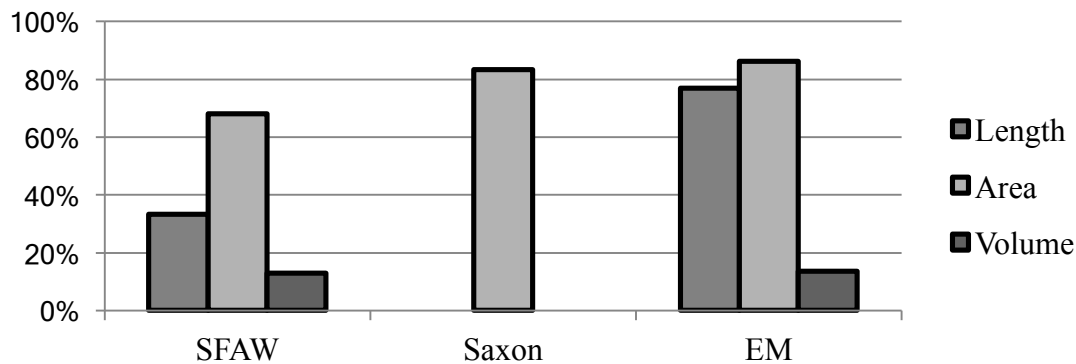


Figure 2. Percentage of DTEs coded as "Exhausting the Space".

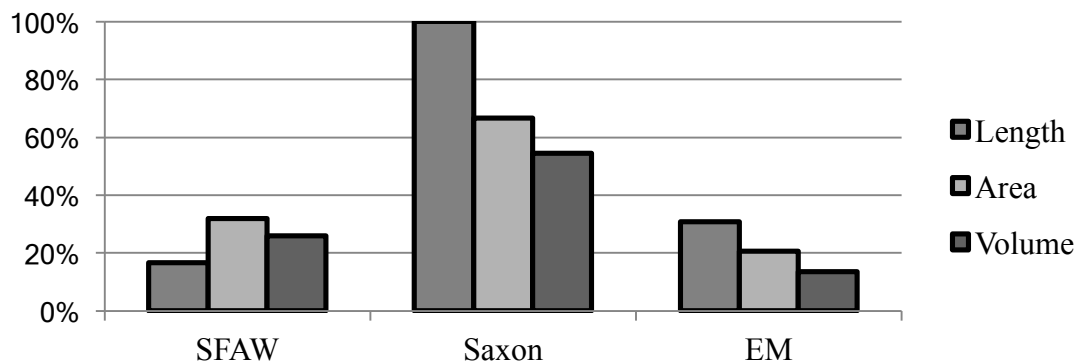


Figure 3: Percentage of DTEs coded for "Quantification".

Length

Across the three curricula, we found 20 DTEs in length measurement in grades K through 3: *SFAW* (6), *Saxon* (1), and *EM* (13). These DTEs appeared more in teacher pages and in the form of declarative statements about measurement (as opposed to questions or problems to pose to the students). The curricular emphasis was in Grades 2 and 3 because 75% of the DTEs appeared in

these grades. Slightly more than half of these DTEs (11 out of 20) were associated with length measurement with specific units. Slightly over half of these DTEs were associated with specific units (11) or covering of space or distance (12).

Area

In area measurement we found 60 DTEs, with *Saxon* (6), *SFAW* (25), and *EM* (29). Of all DTEs, 45% mentioned units, 32% implicitly referenced quantification (e.g., counting), and 78% addressed exhausting space. *Saxon*'s area DTEs mentioned exhausting the space, where its length and volume DTEs did not (see Figure 2). We found a clear distinction in references to units between *EM* ("unit squares") and *SFAW* ("square units").

Volume

Across the three curricula, we coded 131 volume DTEs. Only four appeared in kindergarten and first grade, with a range of 21 to 46 DTEs appearing in Grades 2-5, with higher frequencies in higher grades. Of the 131 DTEs, 24% mentioned units, 22% mentioned quantification, and 12% mentioned exhausting space. One inconsistent characteristic common to all three curricula was in differentiating the meaning of "volume" and "capacity". In some cases, the inconsistency manifested as DTEs that viewed volume and capacity as essentially equivalent and in other places different. As one set of examples, *SFAW* stated in the student materials, "Capacity is the amount a container can hold" (Grade 2, p. 353; Grade 4, p. 609), and later in the teacher materials "Discuss the meaning of the word volume, explaining that it is the amount a container can hold" (Grade 2, p. 69B). Later on students are asked to, "Explain the difference between volume and capacity" (Grade 5, p. 618). Thus, while the terms are seen as essentially the same, it is not clear what the intended differences are in relation to these two terms.

Discussion

The motivation for this study was assessing the extent to which DTEs might contribute to students weak conceptual understanding of measurement. Different from definitions where being minimal is a requirement (Vinner, 1991), our analysis focused on conceptual aspects of definitions because we want to describe how curriculum materials provide a body of knowledge in developing concepts of length, area and volume. We found the lack of consistency in including or excluding aspects of measurement (e.g. units, quantification, exhausting space) in many DTEs, a pattern that may add to students' challenges in learning the meaning of spatial quantities and their measures. For example, if a procedure such as "counting squares" is given in one DTE, and is later omitted in another DTE this raises the question if the student associates a count of squares as a necessary aspect of the definition or simply one of many ways to understand the meaning of area.

Perhaps more troubling is that the inclusion or exclusion of certain conceptual aspects depend (to a great extent) on a particular measure. This may be an example of a lost opportunity for students to see the underlying conceptual structures that exist across measures, making them appear more different than they are similar. At the onset, we hoped to see consistency across particular types of DTEs, but our analysis has shown trends of inconsistency both within curriculum materials, and across measures.

The emphasis (or lack thereof) on certain aspects of measurement in these DTEs might have potential to shape the meaning students make from these definitional expressions. Textbooks have the benefit of providing opportunities to develop students' ideas over extended periods of time, so one might not expect any singular definition to sacrifice readability and meaningfulness to young students for the sake of mathematical precision. This being said, one would hope to see consistency across definitions in relation to their conceptual underpinnings, something we did not find as strongly as we initially expected.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. (REC-0634043 and DRL-0909745).

References

- Bamberger, H. J. & Oberdorf, C. (2010). *Activities to undo math misconceptions, Grades K-2 and Grades 3-5*. Portsmouth, NH: Heinemann.
- Barrett, J. E. & Clements, D. H. (2003). Quantifying path length: Fourth-grade children's developing abstractions for linear measurement. *Cognition and Instruction*, 21(4), 475-520.
- Charles, R., Crown, W., Fennell, F. (2008). *Mathematics, Michigan edition*. Glenview, IL: Pearson Education Inc.
- Gilbertson, N. J. (2015). Attending to precision: Defining quadrilaterals. *Mathematics in Michigan*, 48(2), 14-17.
- Kamii, C., & Kysh, J. (2006). The difficulty of "length 9 width": Is a square the unit of measurement? *Journal of Mathematical Behavior*, 25(2), 105-115.
- Kobiela, M., & Lehrer, R. (2015). The Codevelopment of Mathematical Concepts and the Practice of Defining. *Journal for Research in Mathematics Education*, 46(4), 423-454.
- Larson, N. (2004). *Saxon Math*. Austin, TX: Saxon Publishers, Inc.
- McCrory, R., Francis, A., & Young, S. (2008). *Resource use by instructors of mathematics classes for future elementary teachers*. Monterrey, Mexico: Paper presented at the International Committee on Mathematics Instruction (ICMI-11).
- National Governor's Association - Council of Chief School State Officers. (2010). *Common Core State Standards for Mathematics*. Retrieved from http://corestandards.org/assets/CCSSI_Math%20Standards.pdf.
- Smith III, J. P., Males, L. M., Dietiker, L. C., Lee, K., & Mosier, A. (2013). Curricular Treatments of Length Measurement in the United States: Do They Address Known Learning Challenges?. *Cognition and Instruction*, 31(4), 388-433.
- Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-369). Charlotte, NC: Information Age Publishing.
- Tarr, J. E., Reys, R. E., Reys, B. J., Chavez, O., Shih, J., & Osterlind, S. J. (2008). The impact of middle-grades mathematics curricula and the classroom learning environment on student achievement. *Journal for Research in Mathematics Education*, 39(3), 247-280.
- The University of Chicago School Mathematics Project. (2007). *Everyday Mathematics* (3rd Ed.). Chicago, IL: Wright Group/McGraw Hill.
- Vinner, S. (1991). The role of definitions in learning and teaching mathematics. *Advanced Mathematical Thinking*, 11(II), 65-81.
- Woodward, E. & Byrd, F. (1983). Area: Included topic, neglected concept. *School Science and Mathematics*, 83(4), 343-347.